Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

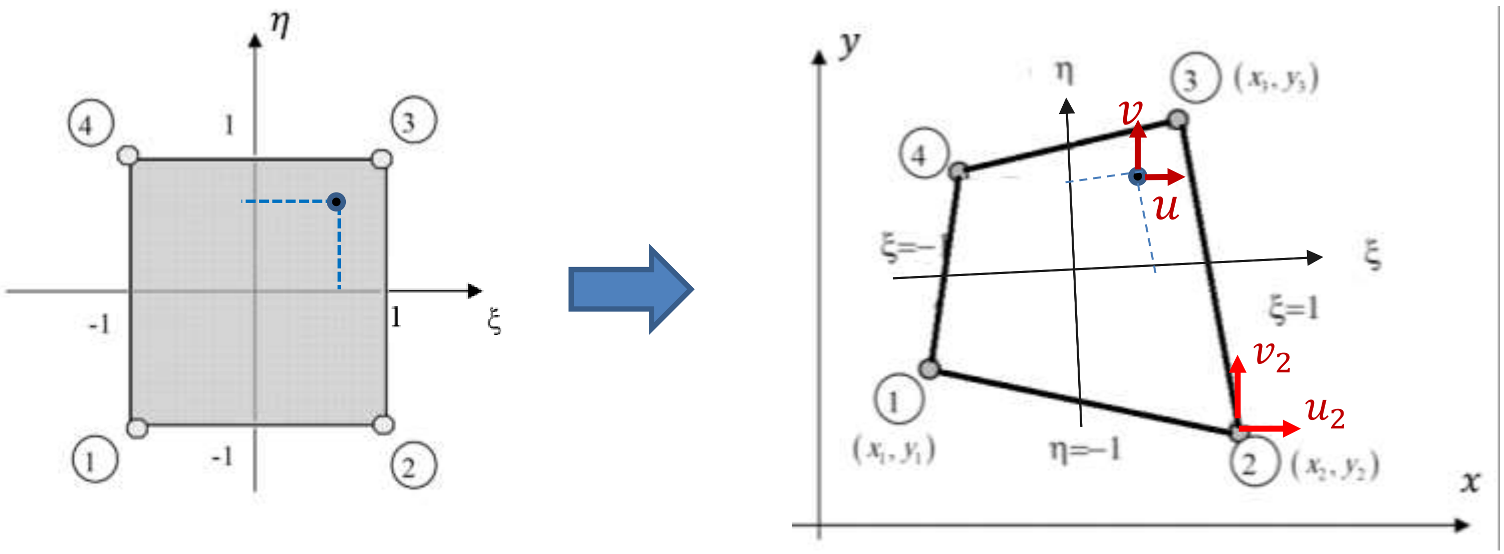
Lecture 6A. 4-node QUAD element

03.2025

4-node QUAD element

natural coordinate system

cartesien coordinate system



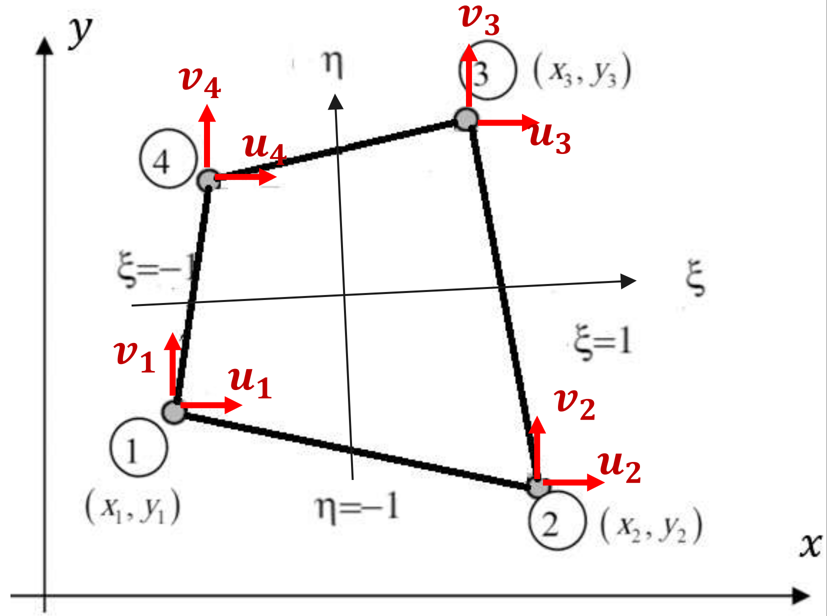
Geometry mapping: $\boxed{(\xi, \eta) \rightarrow (x, y)}$

$(-1,-1) \rightarrow (x_1, y_1)$ $(1,-1) \rightarrow (x_2, y_2)$ $(1,1) \rightarrow (x_3, y_3)$ $(-1,1) \rightarrow (x_4, y_4)$

Isoparametric mapping

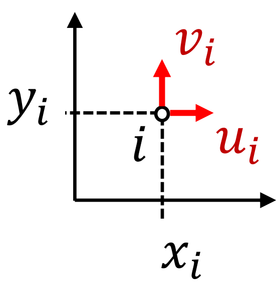
vectors of nodal coordinates

$$\underset{4\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}; \quad \underset{4\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$



$n = 4$; $n_p = 2$ $\rightarrow$ $n_e = n \cdot n_p = 8$

local vector of nodal parameters:



$$\underset{8\times 1}{\{q\}_e} = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_8 \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_4 \end{Bmatrix}_e$$

Isoparametric mapping – the same functions are used to describe the geometry and displacements

Isoparametric mapping

let's construct shape functions:

$$x = a + b\cdot\xi + c\cdot\eta + d\cdot\xi\eta = \lfloor 1, \xi, \eta, \xi\eta \rfloor \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

a, b, c, d – constant

	(ξ,η)		(x,y)	
1	$(-1,-1)$	$\rightarrow$	(x_1,y_1)	$x_1 = 1\cdot a + (-1)\cdot b + (-1)\cdot c + (-1)\cdot(-1)\cdot d$
2	$(1,-1)$	$\rightarrow$	(x_2,y_2)	$x_2 = 1\cdot a + 1\cdot b + (-1)\cdot c + 1\cdot(-1)\cdot d$
3	$(1,1)$	$\rightarrow$	(x_3,y_3)	$x_3 = 1\cdot a + 1\cdot b + 1\cdot c + 1\cdot 1\cdot d$
4	$(-1,1)$	$\rightarrow$	(x_4,y_4)	$x_4 = 1\cdot a + (-1)\cdot b + 1\cdot c + (-1)\cdot 1\cdot d$

$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [A] \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$(\det[A] = -16)$

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e$$

$$x = \lfloor 1, \xi, \eta, \xi\eta \rfloor \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \lfloor 1, \xi, \eta, \xi\eta \rfloor \cdot \underset{4\times 4}{[A]^{-1}} \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e =$$

$$= \lfloor 1, \xi, \eta, \xi \cdot \eta \rfloor \cdot \begin{Bmatrix} \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ -\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ -\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4 \end{Bmatrix} =$$

$$= 1 \cdot \left(\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right) + \xi \cdot \left(-\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4\right) +$$

$$+ \eta \left(-\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right) + \xi\eta \left(\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4\right) =$$

$$= \left(\frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\eta + \frac{1}{4}\xi\eta\right) x_1 + \left(\frac{1}{4} + \frac{1}{4}\xi - \frac{1}{4}\eta - \frac{1}{4}\xi\eta\right) x_2 +$$

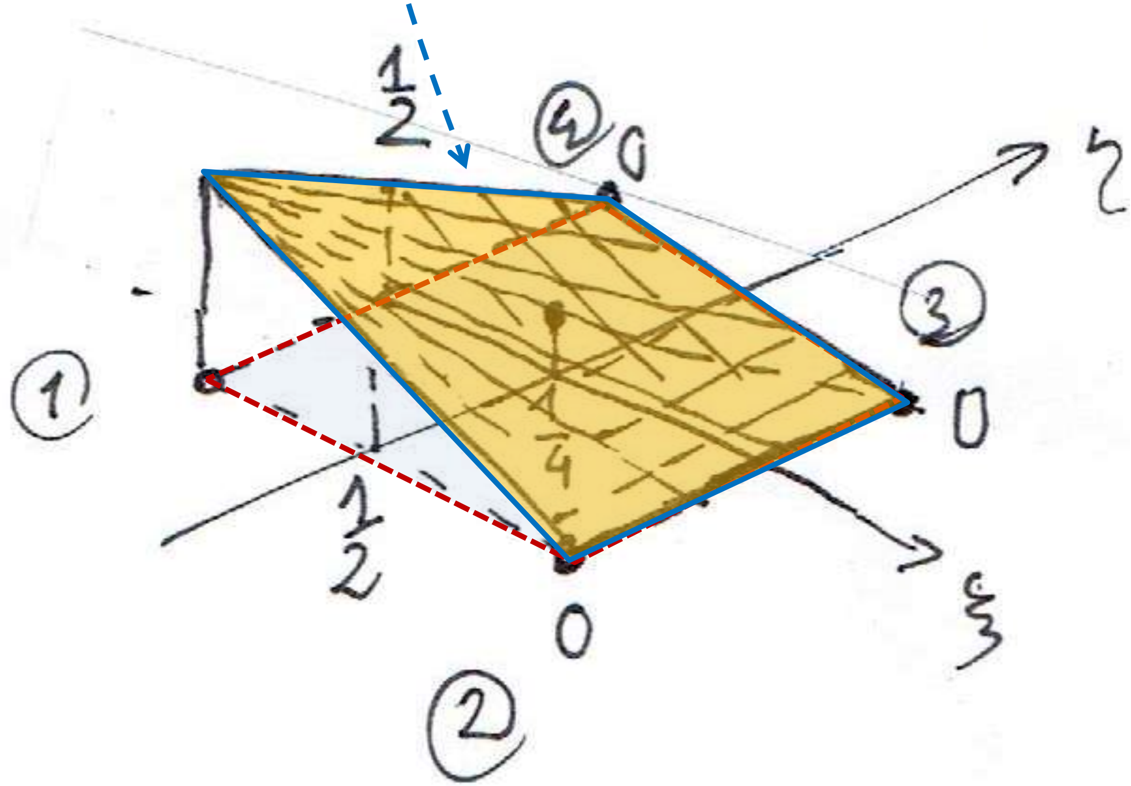
$$+ \left(\frac{1}{4} + \frac{1}{4}\xi + \frac{1}{4}\eta + \frac{1}{4}\xi\eta\right) x_3 + \left(\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\eta - \frac{1}{4}\xi\eta\right) \cdot x_4 =$$

$$= \boxed{\frac{(1-\xi)(1-\eta)}{4}} \cdot x_1 + \boxed{\frac{(1+\xi)(1-\eta)}{4}} \cdot x_2 + \boxed{\frac{(1+\xi)(1+\eta)}{4}} \cdot x_3 + \boxed{\frac{(1-\xi)(1+\eta)}{4}} \cdot x_4 =$$

$$= N_1(\xi, \eta) \cdot x_1 + N_2(\xi, \eta) \cdot x_2 + N_3(\xi, \eta) x_3 + N_4(\xi, \eta) \cdot x_4$$

Shape functions of a 4-node element

$$N_1(\xi,\eta) = ¼(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = ¼(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = ¼(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = ¼(1-\xi)(1+\eta)$$



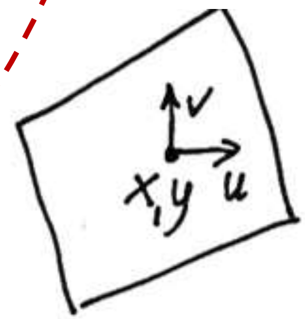
Isoparametric mapping

Isoparametric mapping – the same functions are used to describe the geometry and displacements

$$x = \underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} \cdot \underset{4\times 1}{\{x_i\}_e}$$

$$y = \underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} \cdot \underset{4\times 1}{\{y_i\}_e}$$

$$\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$



position and displacement of any point

where:

$$\underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} = \lfloor N_1, N_2, N_3, N_4 \rfloor$$

$$\underset{2\times 8}{[N(\xi,\eta)]} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

Differential operators in the natural system

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{[J]} \cdot \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$[J]$ – Jacobi matrix

Differential operators in the Cartesian coordinate system

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{\det[J]} \left(\underset{2\times 2}{[J]^{C}}\right)^{T} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\det[J]}\cdot\frac{\partial y}{\partial \eta} & -\frac{1}{\det[J]}\cdot\frac{\partial y}{\partial \xi} \\ -\frac{1}{\det[J]}\cdot\frac{\partial x}{\partial \eta} & \frac{1}{\det[J]}\frac{\partial x}{\partial \xi} \end{bmatrix} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}}$$

$$\det[J] = \boxed{\frac{\partial x}{\partial \xi}} \cdot \boxed{\frac{\partial y}{\partial \eta}} - \boxed{\frac{\partial y}{\partial \xi}} \cdot \boxed{\frac{\partial x}{\partial \eta}} \Rightarrow$$

$$\boxed{\frac{\partial x}{\partial \xi}} = \frac{\partial\left(\underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor} \cdot \underset{4\times 1}{\{x_i\}_e}\right)}{\partial \xi} = \frac{\partial \lfloor N(\xi,\eta)\rfloor}{\partial \xi} \cdot \underset{4\times 1}{\{x_i\}_e} + \underbrace{\frac{\partial \{x_i\}_e}{\partial \xi}}_{=0} \cdot \underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor} =$$

$$= \left\lfloor \frac{\partial N_1}{\partial \xi}, \frac{\partial N_2}{\partial \xi}, \frac{\partial N_3}{\partial \xi}, \frac{\partial N_4}{\partial \xi} \right\rfloor \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e =$$

(discrete values)

$$= \boxed{\left(-\frac{1}{4}(1-\eta)\right) \cdot x_1 + \frac{1}{4}(1-\eta) \cdot x_2 + \frac{1}{4}(1+\eta) x_3 - \frac{1}{4}(1+\eta) x_4}$$

$$\boxed{\frac{\partial y}{\partial \eta}} = \frac{\partial\left(\underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor} \cdot \underset{4\times 1}{\{y_i\}_e}\right)}{\partial \eta} = \frac{\partial \underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor}}{\partial \eta} \cdot \underset{4\times 1}{\{y_i\}_e} =$$

$$= \boxed{\left(-\frac{1}{4}(1-\xi)\right) \cdot y_1 - \frac{1}{4}(1+\xi) \cdot y_2 + \frac{1}{4}(1+\xi) \cdot y_3 + \frac{1}{4}(1-\xi) \cdot y_4}$$

$$\boxed{\frac{\partial y}{\partial \xi}} = \frac{\partial\left(\underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor}\cdot\underset{4\times 1}{\{y_i\}_e}\right)}{\partial \xi} = \frac{\partial \underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor}}{\partial \xi}\cdot\underset{4\times 1}{\{y_i\}}_e =$$

$$= \boxed{\left(-\frac{1}{4}(1-\eta)\right)\cdot y_1 + \frac{1}{4}(1-\eta)\cdot y_2 + \frac{1}{4}(1+\eta)\,y_3 - \frac{1}{4}(1+\eta)\,y_4}$$

$$\boxed{\frac{\partial x}{\partial \eta}} = \frac{\partial\left(\underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor}\cdot\underset{4\times 1}{\{x_i\}_e}\right)}{\partial \eta} = \frac{\partial \underset{1\times 4}{\lfloor N(\xi,\eta)\rfloor}}{\partial \eta}\cdot\underset{4\times 1}{\{x_i\}}_e =$$

$$= \boxed{\left(-\frac{1}{4}(1-\xi)\right)x_1 - \frac{1}{4}(1+\xi)\cdot x_2 + \frac{1}{4}(1+\xi)\,x_3 + \frac{1}{4}(1-\xi)\cdot x_4}$$

$$\boxed{\begin{Bmatrix}\frac{\partial}{\partial x}\\[4pt] \frac{\partial}{\partial y}\end{Bmatrix} = \begin{bmatrix}\frac{1}{\det[J]}\cdot\frac{\partial y}{\partial \eta} & -\frac{1}{\det[J]}\cdot\frac{\partial y}{\partial \xi}\\[4pt] -\frac{1}{\det[J]}\cdot\frac{\partial x}{\partial \eta} & \frac{1}{\det[J]}\cdot\frac{\partial x}{\partial \xi}\end{bmatrix}\cdot\begin{Bmatrix}\frac{\partial}{\partial \xi}\\[4pt] \frac{\partial}{\partial \eta}\end{Bmatrix}}$$

$$\frac{\partial}{\partial x} = \frac{1}{\det[J]}\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial y} = \frac{1}{\det[J]}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi}\right)$$

Gradient matrix for Plain stress or Plain strain condition

$$\underset{3\times 2}{[R(x,y)]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} = \underbrace{\frac{1}{\det[J]} \left[\begin{array}{c:c} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ \hdashline 0 & \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} \\ \hdashline \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{array}\right]}_{\underset{3\times 2}{[R(\xi,\eta)]}}$$

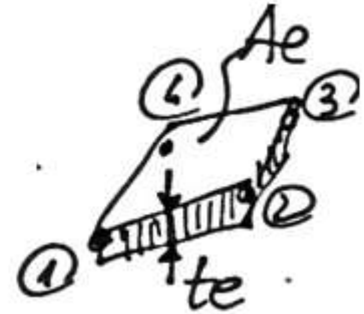
Vector of strain components (Plain stress and Plain strain)

$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 2}{[R(\xi,\eta)]} \cdot \underset{2\times 1}{\{u\}} = \underset{3\times 2}{[R(\xi,\eta)]} \cdot \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$

Vector of stress components (Plain stress and Plain strain)

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$

Elastic strain energy in a finite element



$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2} t_e \int_{A_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \cdot \underset{3\times 1}{\{\sigma\}}\, dA_e =$$

$$= \frac{1}{2} t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor \varepsilon \rfloor \cdot \{\sigma\} \det[J]\, d\xi\, d\eta = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \underset{8\times 8}{[k]_e} \underset{8\times 1}{\{q\}_e}$$

where:

$$[k]_e = t_e \underbrace{\int_{-1}^{1}\int_{-1}^{1} \left(\underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)] \right) d\xi\, d\eta}_{\text{(numerical integration)}}$$

$$[R(x,y)]_{3\times 2} = \frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{bmatrix}$$

$$\underset{3\times 8}{[B(\xi,\eta)]} = \underset{3\times 2}{[R(\xi,\eta)]} \cdot \underset{2\times 8}{[N(\xi,\eta)]} = \begin{bmatrix} r_{11} & r_{12}=0 \\ r_{21}=0 & r_{22} \\ \underset{r_{22}}{\underset{\parallel}{r_{31}}} & \underset{r_{11}}{\underset{\parallel}{r_{32}}} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$r_{11}\cdot N_1 = \frac{\frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta}}{\det[J]} \cdot N_1(\xi,\eta) = \frac{1}{\det[J]}\cdot\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial N_1}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial N_1}{\partial \eta}\right) =$$

$$= \frac{-\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta} + \frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{11} = b_{32}$$

$$r_{11}\cdot N_2 = \frac{\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta} + \frac{1}{4}(1+\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{13} = b_{34}$$

$$r_{11}\cdot N_3 = \frac{\frac{1}{4}(1+\eta)\frac{\partial y}{\partial \eta} - \frac{1}{4}(1+\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{15} = b_{36}$$

$$r_{11}\cdot N_4 = \frac{-\frac{1}{4}(1+\eta)\frac{\partial y}{\partial \eta} - \frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{17} = b_{38}$$

$$\underset{3\times 2}{[R(x,y)]} = \frac{1}{\det[J]}\begin{bmatrix} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ 0 & \boxed{\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi}} \\ \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial y}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{bmatrix}$$

$$r_{22}\cdot N_1 = \frac{\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi}}{\det[J]}\cdot N_1(\xi,\eta) = \frac{1}{\det[J]}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial N_1}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial N_1}{\partial \xi}\right) =$$

$$= \frac{-\frac{1}{4}(1-\xi)\frac{\partial x}{\partial \xi} + \frac{1}{4}(1-\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{22} = b_{31}$$

$$r_{22}\cdot N_2 = \frac{-\frac{1}{4}(1+\xi)\frac{\partial x}{\partial \xi} - \frac{1}{4}(1-\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{24} = b_{33}$$

$$r_{22}\cdot N_3 = \frac{\frac{1}{4}(1+\xi)\frac{\partial x}{\partial \xi} - \frac{1}{4}(1+\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{26} = b_{35}$$

$$r_{22}\cdot N_4 = \frac{\frac{1}{4}(1-\xi)\frac{\partial x}{\partial \xi} + \frac{1}{4}(1+\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{28} = b_{37}$$

$$\underset{3\times 8}{[B]} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} & b_{38} \end{bmatrix}$$

Splitting the elastic strain energy into that due to normal stresses and that due to shear stresses

$$U_e = \frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor \cdot [D] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} d\Omega_e =$$

$$= \underbrace{\frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon_x, \varepsilon_y, 0 \rfloor \underset{3\times 3}{[D]} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix} d\Omega_e}_{U_e^{\sigma} \text{ (normal stress)}} + \underbrace{\frac{1}{2}\int_{\Omega_e} \lfloor 0, 0, \gamma_{xy} \rfloor \cdot \underset{3\times 3}{[D]} \cdot \begin{Bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{Bmatrix} d\Omega_e}_{U_e^{\tau} \text{ (shear stress)}}$$

$$\underset{3\times 1}{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix}} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\{u\}} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{bmatrix}} \cdot \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B_\varepsilon]} \cdot \underset{8\times 1}{\{q\}_e}$$

$$\underset{3\times 1}{\begin{Bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{Bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot \underset{2\times 1}{\{u\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \underset{2\times 8}{[N(\xi,\eta)]} \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B_\gamma]} \cdot \underset{8\times 1}{\{q\}_e}$$

$$[B_\varepsilon] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{18} \\ b_{21} & b_{22} & \cdots & b_{28} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad [B_\gamma] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & \cdots & b_{38} \end{bmatrix}$$

Splitting the elastic strain energy into that due to normal stresses and that due to shear stresses

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\varepsilon}]_e} \underset{8\times 1}{\{q\}_e}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\gamma}]_e} \underset{8\times 1}{\{q\}_e}$$

where:

$$[k_{\varepsilon}]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_{\varepsilon}]^T [D] [B_{\varepsilon}] \cdot \det [J(\xi,\eta)] \right) d\xi \, d\eta$$

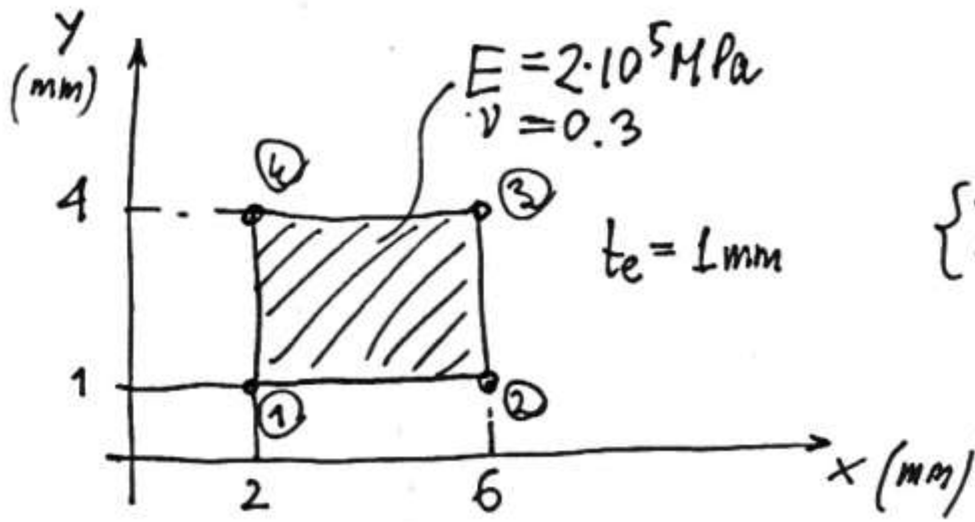
$$[k_{\gamma}]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_{\gamma}]^T [D] [B_{\gamma}] \det [J(\xi,\eta)] \right) d\xi \, d\eta$$

$$[k]_e = [k_{\varepsilon}]_e + [k_{\gamma}]_e$$

Stiffness matrix related to shear strains

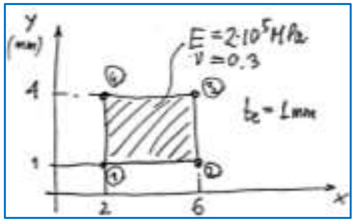
The stiffness matrix related to linear strains

Example 4-node quadrilateral element



$$\{x_i\}_e = \begin{Bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{Bmatrix} ; \quad \{y_i\}_e = \begin{Bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{Bmatrix}$$

$$x_4 = x_1 , \quad x_3 = x_2 , \quad y_2 = y_1 ; \quad y_4 = y_3$$



$$\frac{\partial x}{\partial \xi} = \left(-\frac{1}{4}(1-\eta)\right)\cdot x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4 =$$

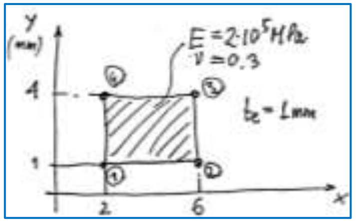
$$= \left(-\frac{1}{4}(1-\eta) - \frac{1}{4}(1+\eta)\right)\cdot x_1 + \left(\frac{1}{4}(1-\eta) + \frac{1}{4}(1+\eta)\right)x_2 =$$

$$= -\frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(6-2) = 2\,mm$$

$$\frac{\partial y}{\partial \eta} = \left(-\frac{1}{4}(1-\xi)\right)\cdot y_1 - \frac{1}{4}(1+\xi)\cdot y_2 + \frac{1}{4}(1+\xi)\cdot y_3 + \frac{1}{4}(1-\xi)\cdot y_4 =$$

$$= \left(-\frac{1}{4}(1-\xi) - \frac{1}{4}(1+\xi)\right)y_1 + \left(\frac{1}{4}(1+\xi) + \frac{1}{4}(1-\xi)\right)y_3 =$$

$$= -\frac{1}{2}y_1 + \frac{1}{2}y_3 = \frac{1}{2}(y_3 - y_1) = \frac{1}{2}(4-1) = 1.5\,mm$$



$$\frac{\partial y}{\partial \xi} = \left(-\frac{1}{4}(1-\eta)\right)\cdot y_1 + \frac{1}{4}(1-\eta)\cdot \underset{}{\overset{\overset{(y_1)}{\parallel}}{y_2}} + \frac{1}{4}(1+\eta)\cdot y_3 - \frac{1}{4}(1+\eta)\cdot \overset{\overset{(y_3)}{\parallel}}{y_4} =$$

$$= 0\cdot y_1 + 0\cdot y_3 = 0$$

$$\frac{\partial x}{\partial \eta} = \left(-\frac{1}{4}(1-\xi)\right)\cdot x_1 - \frac{1}{4}(1+\xi)\cdot x_2 + \frac{1}{4}(1+\xi)\cdot \overset{\overset{(x_2)}{\parallel}}{x_3} + \frac{1}{4}(1-\xi)\cdot \overset{\overset{(x_1)}{\parallel}}{x_4} =$$

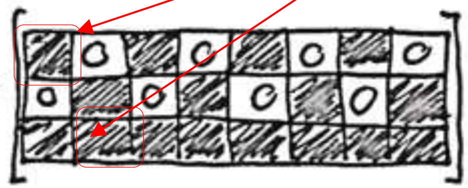
$$= 0\cdot x_1 + 0\cdot x_2 = 0$$

$$\boxed{\det[J] = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial x}{\partial \eta} = 2mm\cdot 1.5mm - 0\cdot 0 = 3mm^2}$$

Strain-displacement matrix:

$$\frac{-\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta}+\frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{11} = b_{32}$$

$$\underset{3\times 8}{[B]} =$$



$$b_{12} = b_{14} = b_{16} = b_{18} = b_{21} = b_{23} = b_{25} = b_{27} = 0$$

$$b_{11} = b_{32} = \frac{-\frac{1}{4}(1-\eta)\cdot 1.5\,mm + \frac{1}{4}(1-\xi)\cdot 0\,mm}{3\,mm^2} = -\frac{1}{8}(1-\eta)\,\frac{1}{mm}$$

$$b_{13} = b_{34} = \frac{\frac{1}{4}(1-\eta)\cdot 1.5\,mm + \frac{1}{4}(1+\xi)\cdot 0\,mm}{3\,mm^2} = \frac{1}{8}(1-\eta)\,\frac{1}{mm}$$

$$b_{15} = b_{36} = \frac{\frac{1}{4}(1+\eta)\cdot 1.5\,mm - \frac{1}{4}(1+\xi)\cdot 0\,mm}{3\,mm^2} = \frac{1}{8}(1+\eta)\,\frac{1}{mm}$$

$$b_{17} = b_{38} = \frac{-\frac{1}{4}(1+\eta)\cdot 1.5\,mm - \frac{1}{4}(1-\xi)\cdot 0\,mm}{3\,mm^2} = -\frac{1}{8}(1+\eta)\,\frac{1}{mm}$$

Strain-displacement matrix:

$$b_{22} = b_{31} = \frac{-\frac{1}{4}(1-\xi)\cdot 2mm + \frac{1}{4}(1-\eta)\cdot 0mm}{3mm^2} = -\frac{1}{6}(1-\xi)\ \frac{1}{mm}$$

$$b_{24} = b_{33} = \frac{-\frac{1}{4}(1+\xi)\cdot 2mm - \frac{1}{4}(1-\eta)\cdot 0mm}{3mm^2} = -\frac{1}{6}(1+\xi)\ \frac{1}{mm}$$

$$b_{26} = b_{35} = \frac{\frac{1}{4}(1+\xi)\cdot 2mm - \frac{1}{4}(1+\eta)\cdot 0mm}{3mm^2} = \frac{1}{6}(1+\xi)\ \frac{1}{mm}$$

$$b_{28} = b_{37} = \frac{\frac{1}{4}(1-\xi)\cdot 2mm + \frac{1}{4}(1+\eta)\cdot 0mm}{3mm^2} = \frac{1}{6}(1-\xi)\ \frac{1}{mm}$$

$$\underset{\substack{3\times 8 \\ \frac{1}{mm}}}{[B(\xi,\eta)]} = \begin{bmatrix} -(1-\eta)/8 & 0 & (1-\eta)/8 & 0 & (1+\eta)/8 & 0 & -(1+\eta)/8 & 0 \\ 0 & -(1-\xi)/6 & 0 & -(1+\xi)/6 & 0 & (1+\xi)/6 & 0 & (1-\xi)/6 \\ -(1-\xi)/6 & -(1-\eta)/8 & -(1+\xi)/6 & (1-\eta)/8 & (1+\xi)/6 & (1+\eta)/8 & (1-\xi)/6 & -(1+\eta)/8 \end{bmatrix}$$

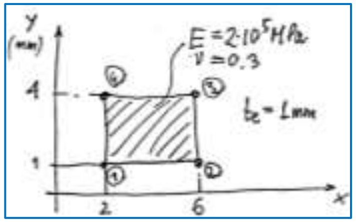
The matrix contains linear terms with respect to the natural coordinates

Strain-displacement matrix :

$$\underset{3\times 8}{\left[B(\xi,\eta)\right]} = \underset{3\times 8}{\left[B_\varepsilon(\xi,\eta)\right]} + \underset{3\times 8}{\left[B_\gamma(\xi,\eta)\right]}$$

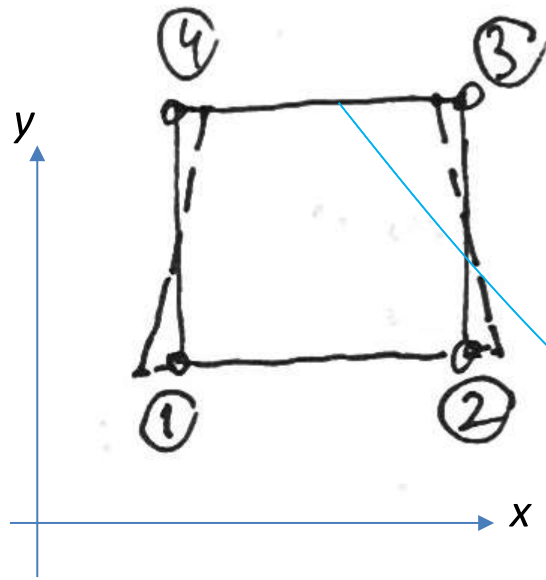
$$\underset{3\times 8}{\left[B_\varepsilon(\xi,\eta)\right]} = \begin{bmatrix} -(1-\eta)/8 & 0 & (1-\eta)/8 & 0 & (1+\eta)/8 & 0 & -(1+\eta)/8 & 0 \\ 0 & -(1-\xi)/6 & 0 & -(1+\xi)/6 & 0 & (1+\xi)/6 & 0 & (1-\xi)/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{mm}$$

$$\underset{3\times 8}{\left[B_\gamma(\xi,\eta)\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\xi)/6 & -(1-\eta)/8 & -(1+\xi)/6 & (1-\eta)/8 & (1+\xi)/6 & (1+\eta)/8 & (1-\xi)/6 & -(1+\eta)/8 \end{bmatrix} \frac{1}{mm}$$



Case 1. *Let's consider the "Bending" deformation in the 4node element*

Vector of nodal parameters:



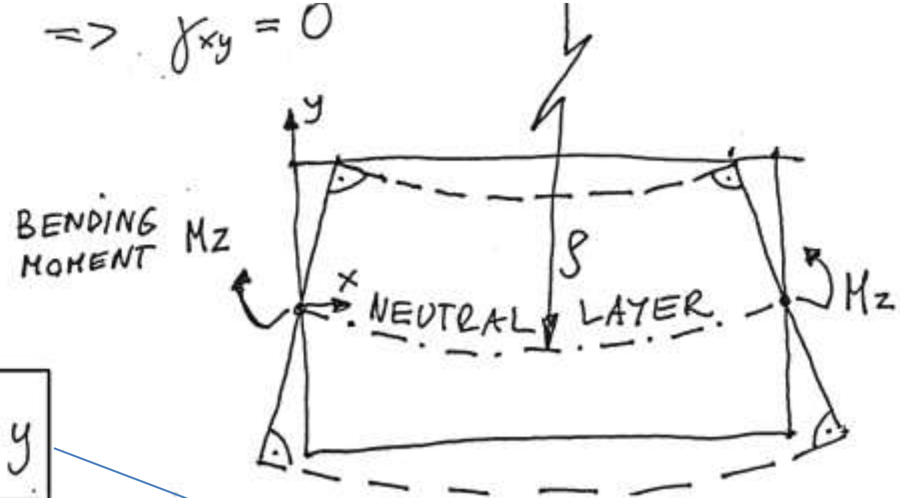
$$\underset{8\times 1\ (mm)}{\{q\}_e} = \begin{array}{c} \left\{\begin{matrix} -0.001 \\ 0 \\ 0.001 \\ 0 \\ -0.001 \\ 0 \\ 0.001 \\ 0 \end{matrix}\right\}_e \end{array} \begin{array}{c} (u_1) \\ \\ (u_2) \\ \\ (u_3) \\ \\ (u_4) \\ \\ \end{array}$$

$$\{x_i\}_e = \begin{Bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{Bmatrix} ; \quad \{y_i\}_e = \begin{Bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{Bmatrix}$$

For the top layer we have: $\varepsilon_x = -\mathbf{0.5 \cdot 10^{-3}}$

<u>Beam theory</u>: Pure beam bending (no shear)

$$\tau_{xy} = 0 \Rightarrow \gamma_{xy} = 0$$



$$\boxed{\varepsilon_x = -\frac{1}{\rho}\cdot y}$$

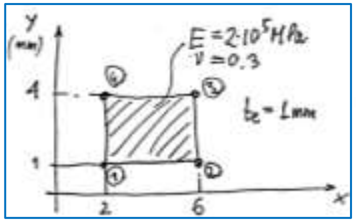
For ***y=1.5*** *mm we have* $\boldsymbol{\varepsilon_x = -0.5\cdot 10^{-3}}$

$$\rho = \frac{1.5}{0.5}\cdot 10^3\,mm = 3000\,mm$$

$$\boxed{\sigma_x = E\,\varepsilon_x} = -\frac{E}{\rho}\cdot y = -\frac{2\cdot 10^5\,MPa}{3000\,mm}\cdot y = -\frac{200}{3}\,\frac{MPa}{mm}\cdot y$$

$$\sigma_x(1.5mm) = -\frac{200}{3}\cdot 1.5\,MPa = -100\,MPa$$

$$\boxed{\sigma_x = -\frac{M_z\cdot y}{J_z}} \Rightarrow -\frac{M_z}{J_z} = -\frac{E}{\rho} \Rightarrow \boxed{\frac{1}{\rho} = \frac{M_z}{E J_z}}$$

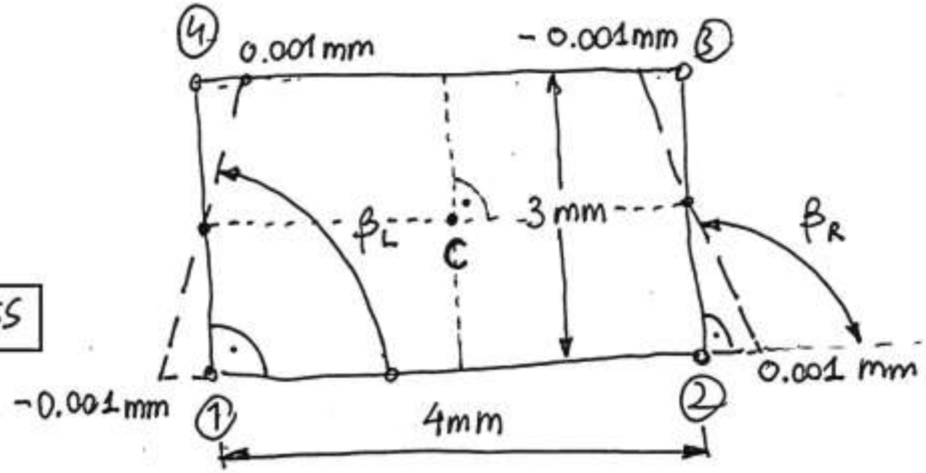


Cont. Case 1. "Bending" in 4-node element

$E = 2\cdot 10^5\,\mathrm{MPa}$

$\nu = 0.3$

$\boxed{\text{PLANE STRESS}}$



Components of strain in the element:

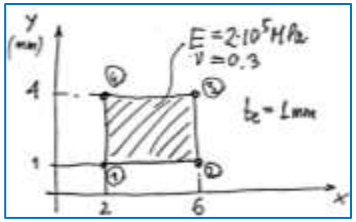
$$\varepsilon_x^{③} = \varepsilon_x^{④} = \frac{\Delta l_{34}}{l_{34}} = \frac{-0.002\,\mathrm{mm}}{4\ \mathrm{mm}} = -0.5\cdot 10^{-3}$$

$$\varepsilon_x^{①} = \varepsilon_x^{②} = \frac{\Delta l_{12}}{l_{12}} = \frac{0.002\ \mathrm{mm}}{4\,\mathrm{mm}} = 0.5\cdot 10^{-3}$$

$$\varepsilon_y^{①} = \varepsilon_y^{②} = \varepsilon_y^{③} = \varepsilon_y^{④} = 0$$

Shear strains!!?

$$\left\{\begin{aligned} \gamma_{xy}^{①} &= \gamma_{xy}^{④} = \frac{\pi}{2} - \beta_L \cong \frac{(0.001-(-0.001))\,\mathrm{mm}}{3\,\mathrm{mm}} = 0.667\cdot 10^{-3} \\ \gamma_{xy}^{③} &= \gamma_{xy}^{②} = \frac{\pi}{2} - \beta_R \cong \frac{(-0.001-0.001)\,\mathrm{mm}}{3\,\mathrm{mm}} = -0.667\cdot 10^{-3} \end{aligned}\right.$$



Stress components in the element:

$$\sigma_x^{(1)} = \sigma_x^{(2)} = \frac{E}{1-\nu^2}\left(\varepsilon_x^{(1)} + \nu\,\varepsilon_y^{(1)}\right) = \frac{2\cdot 10^5\ \text{MPa}}{1-0.3^2}\cdot 0.5\cdot 10^{-3} = 109.89\ \text{MPa}$$

$$\sigma_x^{(3)} = \sigma_x^{(4)} = \frac{E}{1-\nu^2}\left(\varepsilon_x^{(3)} + \nu\,\varepsilon_y^{(3)}\right) = -109.89\ \text{MPa}$$

It came out a little more

Stresses in Y direction!!?

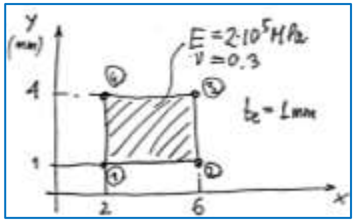
$$\sigma_y^{(1)} = \sigma_y^{(2)} = \frac{E}{1-\nu^2}\left(\varepsilon_y^{(1)} + \nu\,\varepsilon_x^{(1)}\right) = \frac{2\cdot 10^5\ \text{MPa}}{1-0.3^2}\cdot 0.3\cdot 0.5\cdot 10^{-3} = 32.97\ \text{MPa}$$

$$\sigma_y^{(3)} = \sigma_y^{(4)} = \frac{E}{1-\nu^2}\left(\varepsilon_y^{(3)} + \nu\,\varepsilon_y^{(3)}\right) = -32.97\ \text{MPa}$$

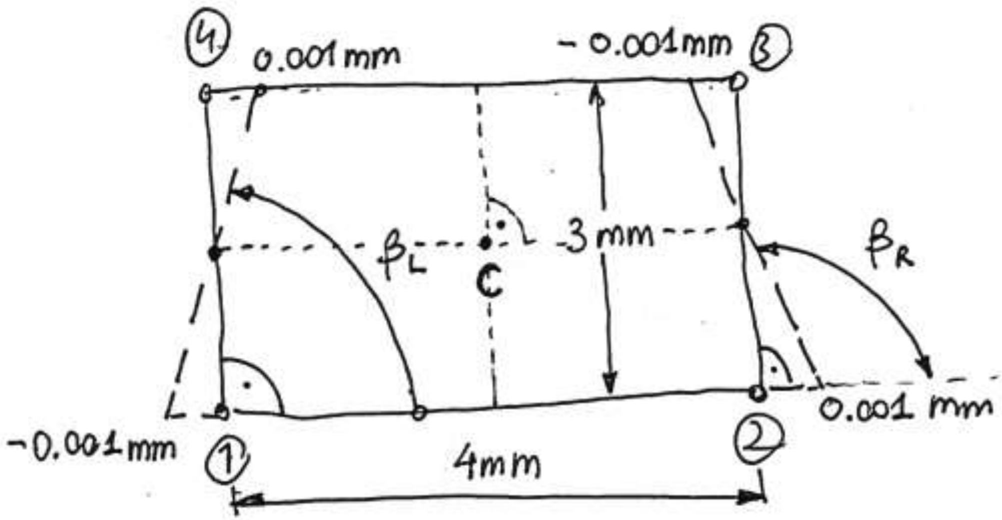
Shear stress !!?

$$\tau_{xy}^{(1)} = \tau_{xy}^{(4)} = \gamma_{xy}^{(1)}\cdot G = 0.667\cdot 10^{-3}\cdot \frac{2\cdot 10^5\ \text{MPa}}{2(1+0.3)} = 51.28\ \text{MPa}$$

$$\tau_{xy}^{(2)} = \tau_{xy}^{(3)} = \gamma_{xy}^{(2)}\cdot G = -51.28\ \text{MPa}$$



Strain and stress components at the center (point C):

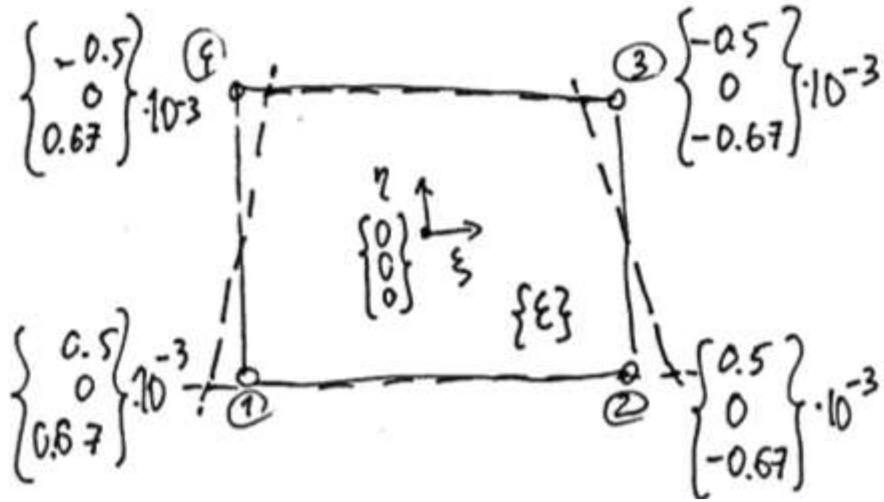


$$\varepsilon_x^c = 0, \quad \varepsilon_y^c = 0, \quad \gamma_{xy}^c = 0 \quad \Rightarrow \quad \sigma_x^c = 0, \quad \sigma_y^c = 0, \quad \tau_{xy}^c = 0$$

There is no strain or stress at the center point!

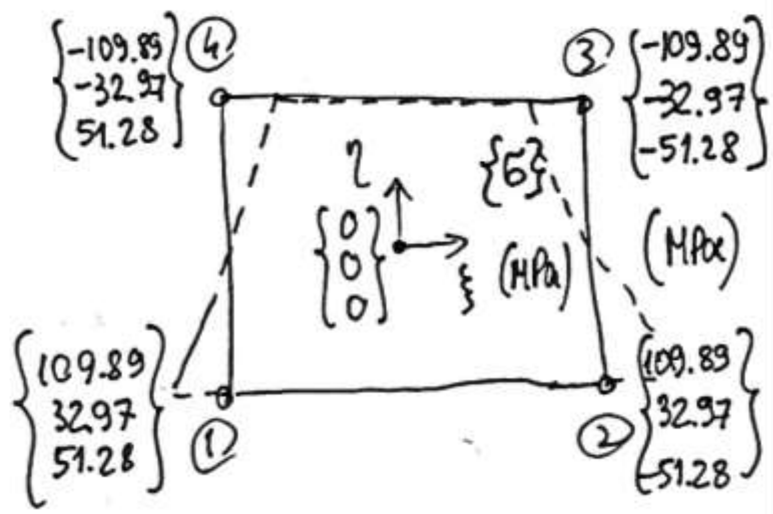
If we calculate the strains in an element from the strain-displacement matrix:

$$\underset{3\times1}{\{\varepsilon\}} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \underset{3\times8}{[B(\xi,\eta)]}\cdot \underset{8\times1}{\{q\}_e}$$



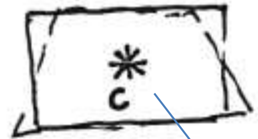
We will calculate the components of the stress state using the matrix of elastic constants:

$$\underset{3\times1}{\{\sigma\}} = \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon\}} = \frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&\frac{1}{2}(1-\nu)\end{bmatrix}\cdot\underset{3\times1}{\{\varepsilon\}}$$



Let’s use numerical integration with one Gaussian point

Numerical integration n=1



$W_1 \cdot W_1 = 4$

Let's calculate the elastic strain energy in the element:

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \underset{8\times 8}{[k]_e} \cdot \underset{8\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \int_{\Omega_e} \underset{8\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B]} \, d\Omega_e \cdot \underset{8\times 1}{\{q\}_e} =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)] \, d\xi \, d\eta \cdot \underset{8\times 1}{\{q\}_e} = \left| \begin{matrix} n=1 \\ \xi_1 = 0 \\ \eta_1 = 0 \end{matrix} \right| =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \, t_e \cdot \underset{8\times 3}{[B(0,0)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(0,0)]} \cdot \det[J(0,0)] \cdot W_1 \cdot W_1 \cdot \underset{8\times 1}{\{q\}_e} \boxed{=0}$$

Energy comes out 0 !

$$\frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} = 2\text{mm} \cdot 1.5\text{mm} - 0 \cdot 0 = 3\,\text{mm}^2$$

$$\underset{3\times 3}{[D]} = \frac{2 \cdot 10^5}{1 - 0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-0.3) \end{bmatrix} \text{MPa}$$

$$\underset{3\times 8}{[B(0,0)]} = \begin{bmatrix} -1/8 & 0 & 1/8 & 0 & 1/8 & 0 & -1/8 & 0 \\ 0 & -1/6 & 0 & -1/6 & 0 & 1/6 & 0 & 1/6 \\ -1/6 & -1/8 & -1/6 & 1/8 & 1/6 & 1/8 & 1/6 & -1/8 \end{bmatrix} \frac{1}{\text{mm}}$$

Let's try to calculate the elastic strain energy of the element again in a different way:

$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \cdot \underset{3\times 1}{\{\sigma\}} \, d\Omega_e = \frac{1}{2} t_e \int_{-1}^{1}\int_{-1}^{1} \underset{1\times 3}{\lfloor \varepsilon(\xi,\eta) \rfloor} \cdot \underset{3\times 1}{\{\sigma(\xi,\eta)\}} \cdot \det[J(\xi,\eta)] \cdot d\xi \, d\eta =$$

$$= \left| \begin{array}{l} n = 1 \\ \xi_1 = 0 \\ \eta_1 = 0 \\ W_1 W_1 = 4 \end{array} \right| = \frac{1}{2} t_e \underset{1\times 3}{\lfloor \varepsilon(0,0) \rfloor} \cdot \underset{3\times 1}{\{\sigma(0,0)\}} \cdot \det[J(0,0)] \cdot W_1 W_1 = \boxed{0}$$

It's zero again!

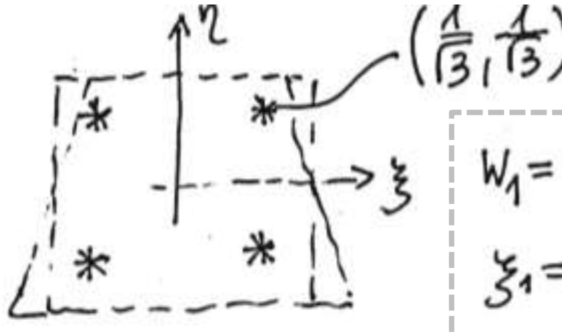
$$\lfloor \varepsilon_x^c, \varepsilon_y^c, \gamma_{xy}^c \rfloor = \lfloor 0, 0, 0 \rfloor \qquad \begin{Bmatrix} \sigma_x^c \\ \sigma_y^c \\ \tau_{xy}^c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{U_e^{\tau} = 0 \, , \; U_e^{\sigma} = 0}$$

As we can see, the elasticstrain energy due to normal and shear stress is zero.

We use numerical integration with two Gaussian points

Numerical integration n=2



$W_1 = W_2 = 1$

$\xi_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}}$

$\eta_1 = -\frac{1}{\sqrt{3}}, \quad \eta_2 = \frac{1}{\sqrt{3}}$

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta \cdot \underset{8\times 1}{\{q\}_e} =$$

$$\underset{8\times 8}{[f(\xi,\eta)]} = \underset{8\times 3}{[B(\xi,\eta)]^T} \cdot \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(\xi,\eta)]} \cdot \det[J(\xi,\eta)]$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 8}{[f(\xi,\eta)]}\, d\xi\, d\eta \cdot \underset{8\times 1}{\{q\}_e} =$$

It came out differently!

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} t_e \cdot \left(\underset{8\times 8}{[f(\xi_1,\eta_1)]} \cdot W_1 W_1 + \underset{8\times 8}{[f(\xi_2,\eta_1)]} \cdot W_2 W_1 + \underset{8\times 8}{[f(\xi_2,\eta_2)]} \cdot W_2 W_2 + \underset{8\times 8}{[f(\xi_1,\eta_2)]} \cdot W_1 W_2 \right) \cdot \underset{8\times 1}{\{q\}_e} = \boxed{0.1783 \text{ Nmm}}$$

Let's try to calculate the elastic strain energy of the element again differently (separately for normal stress and separately for shear stress):

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \cdot \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B_\varepsilon(\xi,\eta)]^T} \cdot \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B_\varepsilon(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta \,\{q\}_e =$$

$$= 0.1099 \text{ Nmm}$$ *Elastic strain energy due to normal stress*

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \cdot \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B_\gamma(\xi,\eta)]^T} \cdot \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B_\gamma(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta \cdot \{q\}_e =$$

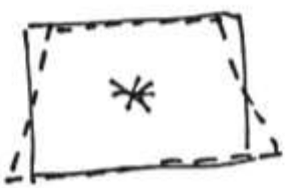
$$= 0.0684 \text{ Nmm}$$

Elastic strain energy due to shear stress

Elastic strain energy *(comparison for different numbers of integration points)*

Numerical integration

$n = 1$



$w_1 w_1 = 4$

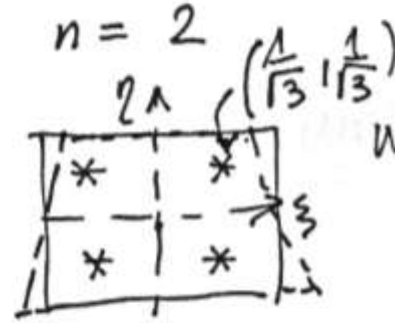
$$U_e = \tfrac{1}{2} L \underset{1\times 8}{\{q\}_e^T} \underset{8\times 8}{[k]_e} \cdot \underset{8\times 1}{\{q\}_e} = 0$$

$$U_e^{\sigma} = \tfrac{1}{2} L \underset{1\times 8}{\{q\}_e^T} \underset{8\times 8}{[k_{\varepsilon}]_e} \underset{8\times 1}{\{q\}_e} = 0$$

$$U_e^{\tau} = \tfrac{1}{2} L \underset{1\times 8}{\{q\}_e^T} \underset{8\times 8}{[k_{\gamma}]} \{q\}_e = 0$$

Zero energy mode („hourglassing”)

$n = 2$



$W_1 = W_2 = 1$

$$U_e = 0{,}1783 \quad Nmm$$

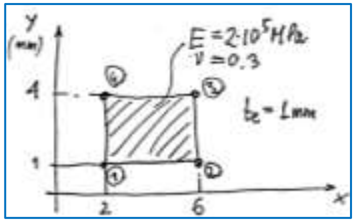
$$U_e^{\sigma} = 0{,}1099 \quad Nmm$$

$$U_e^{\tau} = \boxed{0{,}0684 \quad Nmm}$$

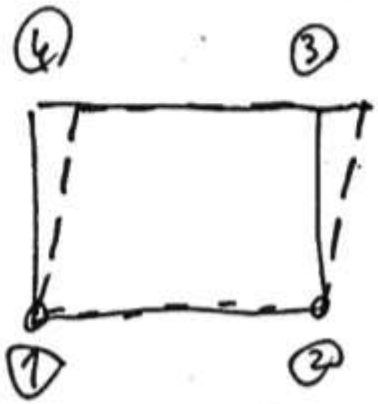
Nonzero shear stress energy!

$$\left(U_e^{\tau} = 38\% \cdot U_e \ ? \right)$$

The element is more rigid („shear locking”)



***Case 2**. Let's consider the "Shear" deformation*



$$\underset{\substack{8\times 1 \\ (mm)}}{\{q\}_e} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.001 \\ 0 \\ 0.001 \\ 0 \end{Bmatrix} \begin{matrix} \\ \\ \\ \\ (u_3) \\ \\ (u_4) \\ \\ \end{matrix}$$

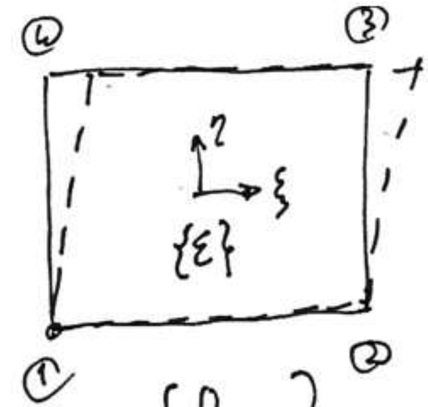
Estimation of strain and stress state:

$$\gamma_{xy} \cong \frac{0.001\,mm}{l_{23}} = \frac{0.001\,mm}{3\ mm} = 0.33 \cdot 10^{-3}$$

$$\tau_{xy} = G \cdot \gamma_{xy} = \frac{E}{2(1+\nu)} \cdot \gamma_{xy} = 25.64\,MPa$$

Let's calculate the strains in the element from the strain-displacement matrix:

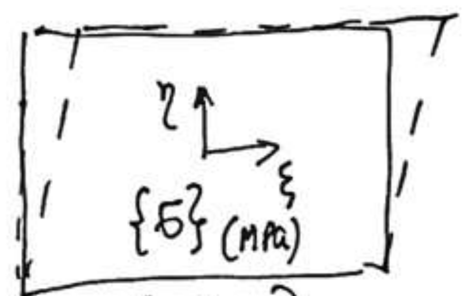
$$\underset{3\times1}{\{\varepsilon\}} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \underset{3\times8}{[B(\xi,\eta)]}\cdot \underset{8\times1}{\{q\}_e}$$



$$\begin{Bmatrix}0\\ 0\\ 0.33\cdot10^{-3}\end{Bmatrix}$$ - *constant*

We will calculate the components of the stress state using the matrix of elastic constants:

$$\underset{3\times1}{\{\sigma\}} = \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon\}} = \frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&\frac{1}{2}(1-\nu)\end{bmatrix}$$

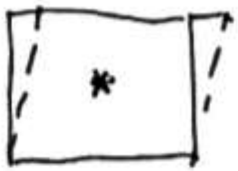


$$\begin{Bmatrix}0\\ 0\\ 25.641\end{Bmatrix}$$ - *constant*

Elastic strain energy *(comparison for different numbers of integration points)*

Numerical integration

$n = 1$



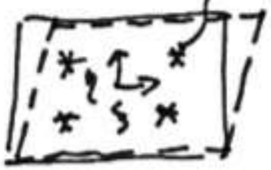
$W_1 W_1 = 4$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e \cdot [k]_e \cdot \{q\}_e = 0.0513\ \text{Nmm}$$

$$U_e^{\sigma} = 0 \quad , \quad U_e^{\tau} = U_e$$

$n = 2$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



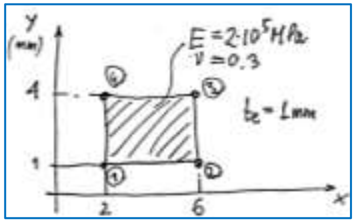
$w_1 = W_2 = 1$

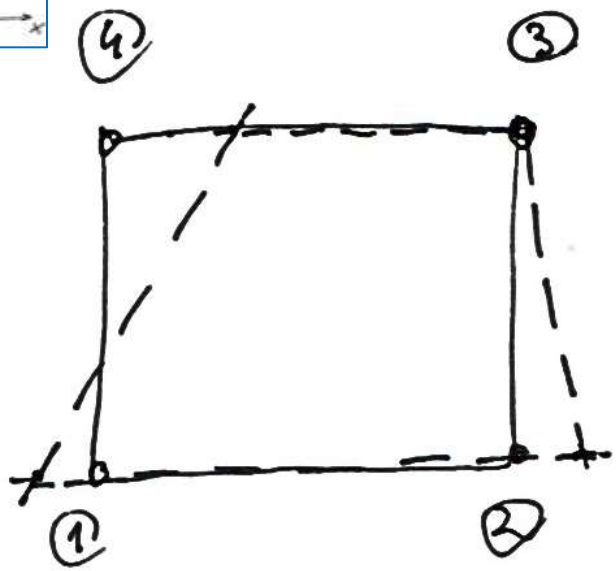
$$U_e = 0.0513\ \text{Nmm}$$

$$U_e^{\sigma} = 0 \quad , \quad U_e^{\tau} = U_e$$

The value is identical regardless of the number of Gauss points

Case 3. *Let's consider the deformation "Bending + Shear"(superposition 1+2)*



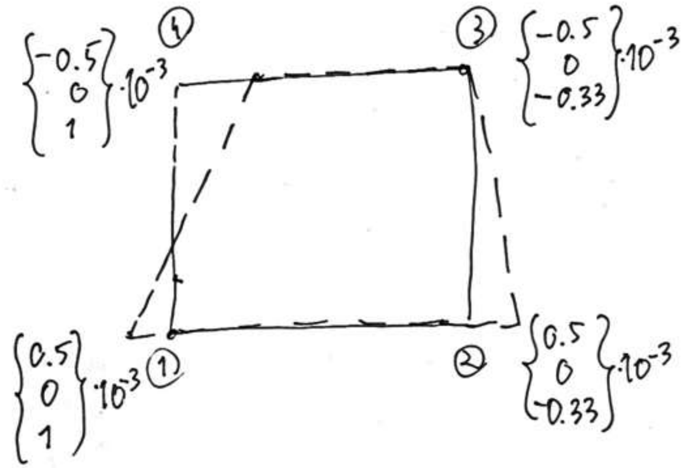


$$\{q\}_{e\,(8\times1)} = \begin{array}{c} \left\{ \begin{array}{c} -0.001 \\ 0 \\ 0.001 \\ 0 \\ 0 \\ 0 \\ 0.002 \\ 0 \end{array} \right\} \end{array} \begin{array}{l} (u_1) \\ \\ (u_2) \\ \\ \\ \\ (u_4) \\ \\ \end{array}$$

(mm)

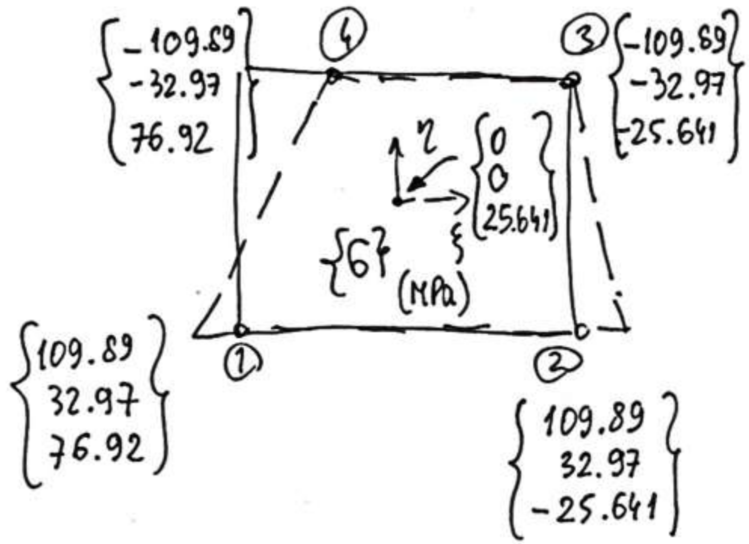
Let's calculate the strains in the element from the strain-displacement matrix:

$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 8}{[B(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$



We will calculate the components of the stress state using the matrix of elastic constants:

$$\underset{3\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$



Elastic strain energy *(comparison for different numbers of integration points)*

Numerical integration

$n = 1$

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k]_e} \underset{8\times 1}{\{q\}_e} = \underset{Nmm}{0.0513}$$

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\varepsilon}]} \{q\}_e = \underset{Nmm}{0}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} [k_{\gamma}] \{q\}_e = 0.0513\,Nmm = U_e$$

$n = 2$

$$U_e = 0.22955\ Nmm$$

$$U_e^{\sigma} = 0.1099\ Nmm$$

$$U_e^{\tau} = 0.1197\ Nmm$$

$$U_e^{\tau} = 52\%\ U_e$$

$$U_e^{\tau}(case3) = U_e^{\tau}(case1) + U_e^{\tau}(case2) =$$

$$= (0.0684 + 0.0513)\ Nmm = 0.1197\ Nmm$$

„shear locking”

Summary

CASE U_e [Nmm]	n=1			n=2		
	U_e^σ	U_e^τ	U_e	U_e^σ	U_e^τ	U_e
1. „BENDING"	0	0	0	0.1099	0.0684	0.1783
2. „SHEAR"	0	0.0513	0.0513	0	0.0513	0.0513
3. „BENDING + SHEAR"	0 = (0+0)	0.0513 = (0+0.0513)	0.0513 (0+0.0513)	0.1099 = (0 + 0.1099)	0.1197 = (0.0684 + 0.0513)	0.22955 = 0.0513 + 0.1783

„hourglassing”

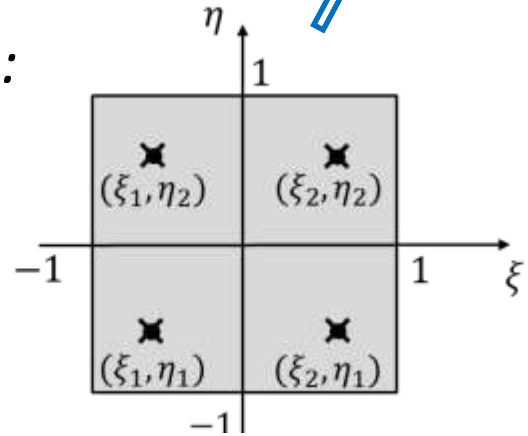
„shear locking”

What to do to improve results?

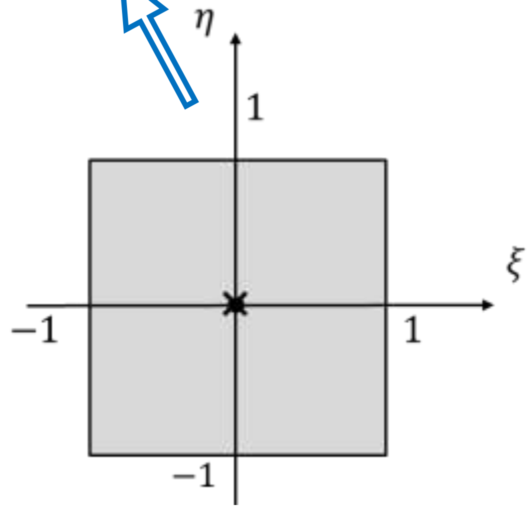
$$[k]_e = [k_\varepsilon]_e + [k_\gamma]_e$$

Conslusions:

(element technology)



Full integration



Reduced integration

$$[k_\varepsilon]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_\varepsilon]^T [D] [B_\varepsilon] \cdot \det[J(\xi,\eta)]\right) d\xi\, d\eta$$

$$[k_\gamma]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_\gamma]^T [D] [B_\gamma] \det[J(\xi,\eta)]\right) d\xi\, d\eta$$

Mixed quadrature rule

Full integration ($n = 2$) :

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\varepsilon}]} \{q\}_e = 0.1099\ Nmm$$

Reduced integration ($n = 1$) :

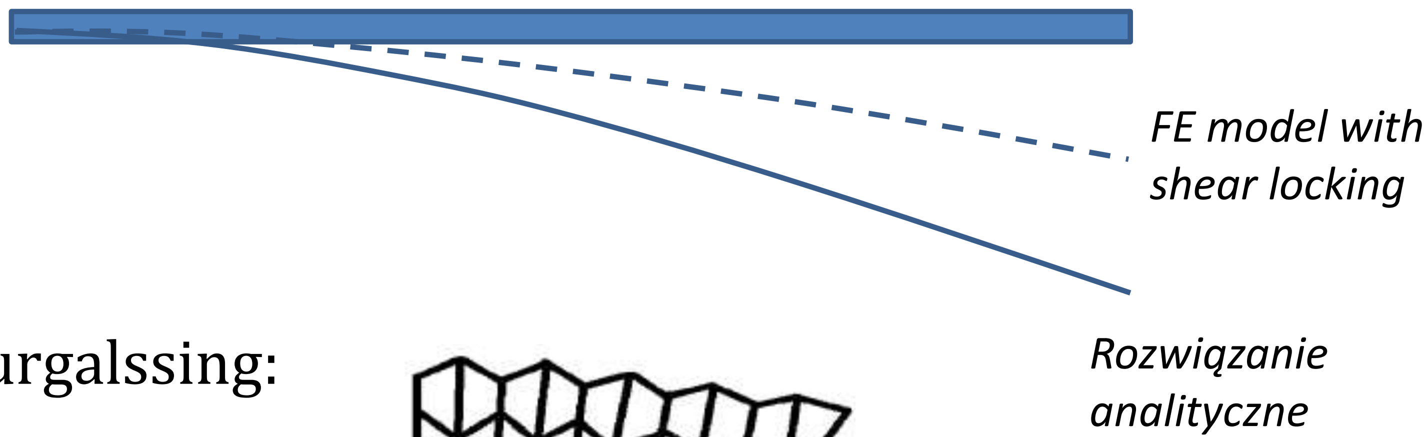
$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} [k_{\gamma}] \{q\}_e = 0.0513\ Nmm$$

$$= U_e$$

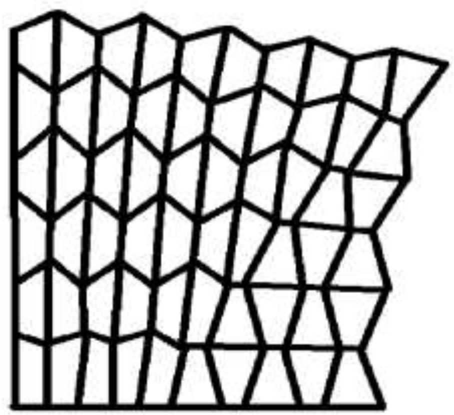
$$U_e = U_e^{\sigma} + U_e^{\tau} = 0.16117\ Nmm$$

$$U_e\,(case3) = U_e^{\sigma}(case1) + U_e^{\tau}(case2)$$

– shear locking:



– hourgalssing:



– volumetric locking in nearly incompressible materials ($\nu \cong 0.5$)

Element Technology – Linear Materials

Element	Stress State	Poisson's ratio <= 0.49	Poisson's ratio > 0.49 (or anisotropic materials)
PLANE182	Plane stress	KEYOPT(1) = 2 (Enhanced strain formulation)	KEYOPT(1) = 2 (Enhanced strain formulation)
	Not plane stress	KEYOPT(1) = 3 (Simplified enhanced strain formulation)	KEYOPT(1) = 2 (Enhanced strain formulation)
PLANE183	Plane stress	No change	No change
	Not plane stress	No change	No change
SOLID185		KEYOPT(2) = 3 (Simplified enhanced strain formulation)	KEYOPT(2) = 2 (Enhanced strain formulation)
SOLID186		KEYOPT(2) = 0 (Uniform reduced integration)	KEYOPT(2) = 0 (Uniform reduced integration)
SHELL281		No change	No change

(+additional shape features)

Shear Locking and Hourglassing in MSC Nastran, ABAQUS, and ANSYS

Eric Qiuli Sun

Abstract

A solid beam and a composite beam were used to compare how MSC Nastran, ABAQUS, and ANSYS handled the numerical difficulties of shear locking and hourglassing. Their tip displacements and first modes were computed, normalized, and listed in multiple tables under various situations. It was found that fully integrated first order solid elements in these three finite element codes exhibited similar shear locking. It is thus recommended that one should avoid using this type of element in bending applications and modal analysis. There was, however, no such shear locking with fully integrated second order solid elements. Reduced integration first order solid elements in ABAQUS and ANSYS suffered from hourglassing when a mesh was coarse. If there was only one layer of elements, the reported first mode of the beam examples from ABAQUS and ANSYS was excessively smaller than the converged solutions due to hourglassing. At least four layers of elements should, therefore, be used in ABAQUS and ANSYS. MSC Nastran outperformed ABAQUS and ANSYS by virtually eliminating the annoying hourglassing of reduced integration first order 3D solid elements because it employed bubble functions to control the propagation of non-physical zero-energy modes. Even if there was only one layer of such elements, MSC Nastran could still manage to produce reasonably accurate results. This is very convenient because it is much less prone to errors when using reduced integration first order 3D solid elements in MSC Nastran.

https://moodle.umontpellier.fr/pluginfile.php/480056/mod_resource/content/0/Sun-ShearLocking-Hourglassing.pdf